ESKELSON

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ON ANALYTIC CONTINUATION

BY

OLA MATTIE JOSEPHINE ESKELSON

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James Byrnie Shaw In Charge of Major Work

J. Townsend Head of Department

Recommendation concurred in:

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CHAPTER I.

INTRODUCTION.

1. *Historical.* In his book *Prolongement Analytique** Zoretti says. "Cauchy should be considered as the creator of the theory of analytic functions of a complex variable. It is to him that we owe the introduction into science of the notion of a holomorphic function and the possibility of its development in a Taylor's series. The notion of radius of convergence and singular points follows naturally from this."

It is due to the integral formula of Cauchy and the theory based upon it that the above statements were made. Weierstrass uses an integral power series to represent an analytic function and defines the analytic function in its entirety as the aggregate of all the possible analytic continuations of some one series.°

Mittag-Leffler, Lindelöf and Schwarz have given us methods of analytic continuation which lead to other expressions than power series for the function in its region of existence.

2. The aim of the first chapter of this thesis is to define analytic functions sufficiently to make the remaining chapters understood. The second chapter contains the methods of analytic continuation worked out by Weierstrass and Mittag-Leffler. The third chapter contains the methods worked out by Schwarz and Lindelöf which depend on linear substitutions leading to conformal transformation. The fourth chapter contains some methods of a formal character. At the close of the thesis will be found

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* p. 1.

° Harkness and Morley, *Introduction to Analytic Functions*, p. 105.

a bibliography of the books and memoirs consulted in preparing the thesis, and some others for reference.

3. *Definitions of Function.* If x and y are two real variables so related that when we give to x a set of values, y takes a corresponding set or sets of values, we say that y is a function of x. This is the most general and elementary idea of function.*

4. The set of values which the independent variable takes is called the range of the independent variable, and a corresponding set of values which the function $y=f(x)$ takes is called the range of the function values for one branch of the function. For example in the function

$$f(x) = \sin x \qquad (1)$$

the values of $f(x)$ range between -1 and $+1$ while the values of x range between $-\infty$ and $+\infty$. If

$$f(x) = 1 + x + x^2 + \ldots\ldots \qquad (2)$$

the values of $f(x)$ range between $\frac{1}{2}$ and $+\infty$ and the values of x range between -1 and $+1$. In (1) the domain of the function values is limited, while in (2) it is unlimited; in (1) the domain of the independent variable values is unlimited, while in (2) it is partially limited.

5. Let us consider the common defining expressions for a function of one real variable, since the same methods of definition often apply also to functions of a complex variable. By an analytic expression is meant an expression made up of the variable quantity and constants by combining them by means of addition, subtraction, multiplication, division, involution, evolution, and taking of limits.[o] A function may be de-

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* In this thesis x and y will always be used to denote real variables and z and w to denote complex variables.

[o] Encyclopédie des Sciences Mathématiques, vol. 2, part 1, 1. pp. 6-7.

fined by a finite algebraic expression such as $y = x^2$ (1). It is also defined by a series of powers of the variable as

$$f(x) = 1 + x + x^2 + \ldots\ldots + x^n + \ldots. \qquad (2)$$

or by a series of rational functions in the variable such as*

$$f(x) = \sum_{k=1}^{k=n} G_k\left(\frac{1}{x-a_k}\right) \qquad (3)$$

where $G_k\left(\frac{1}{x-a_k}\right)$ denotes an infinite series and a_k constants.

A function may be defined by a series of polynomials, as

$$f(x) = \alpha_0 + \alpha_1[g_1(x)] + \ldots\ldots + \alpha_n[g_n(x)] + \ldots\ldots \qquad (4)$$

where $g_n(x) = a_0 + a_1x + \ldots\ldots + a_nx^n$ is some polynomial in which the a's are constants and n is always a positive integer or zero. A function may be defined by a series of continuous functions, as

$$f(x) = \beta_0 + \beta_1[D_1(x)] + \ldots\ldots + \beta_n[D_n(x)] + \ldots. \qquad (5)$$

where we know $D_i(x)$ to be some continuous function.

A single analytic expression may represent within divers intervals distinct algebraic or transcendental functions; being in the sense of Euler one and the same function; thus if°

$$f(x) = \underset{n=\infty}{L} \frac{x^{2(n+1)} + \log(x^2) + 1}{x^{2n} + 1} \qquad (6)$$

we have $f(x) = \log(x^2) + 1$ when $|x|<1$

$f(x) = 1$ when $x = \pm 1$

$f(x) = x^2$ when $|x|>1$.

One and the same function, according to Euler, may then in passing from one interval to another change completely in the character of its properties, although the expression remains the same. The usage has of late been such as to call these functions of Euler's expressions, the meaning of function being different. A function may be defined by a definite in-

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* *Acta Mathematica*, vol. 1(1882), p. 112, by Mittag-Leffler.

° *Encyclopédie des Sciences Mathématiques*, vol. 2, part 1, 1. p. 8.

tegral, as

$$\int_0^\pi \cos(tx)dx \qquad \int_0^2 (1-x^2t^2)dx \qquad (7).$$

6. A function may also be defined by a functional relation with restrictive conditions, thus

$$\Gamma(x+1) = x\Gamma(x) \qquad (8).$$

7. In considering a function of a complex variable we note that the range defined by a function of a complex variable is different from that defined by a function of a real variable.

By degree of freedom I mean the number of dimensions in which the independent variable can vary. The function $f(x)$ of a real variable has but one degree of freedom since x can vary in only one dimension; while the function $f(z) = f(x+iy)$ of a complex variable has two degrees of freedom because the variable $z=x+iy$ varies in two dimensions since x varies in one dimension and y varies in another dimension and the variation of each is independent of that of the other. The boundary of the function $f(x)$ is determined by the two extreme points of the curve it defines, while the boundary of the values of the function $f(z)$ consists of the set of points in the w-plane making up the curve enclosing the region defined by $f(z)$.

Various definitions of function, with or without descriptive adjectives have been given. Some of these are quoted below. The following is Riemann's definition of a function of a complex variable. A complex quantity w is a function of another complex quantity z when they change together in such a manner that the value of $\frac{dw}{dz}$ is independent of the value of the differential element dz.*

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* *Theory of Functions*, Forsyth , p.8.

The meaning of the above definition is that $\frac{dw}{dz}$ must be independent of the argument of Δz.

The defining expressions for functions of a complex variable are formally similar to those of a real variable. The difference is in the multiplication formulae of complex numbers.

Analytic Function is defined thus: A function $f(z)$ is called an analytic function of z in the given region S, if it is fully defined for that region and has at each point a derivative.*

Monogenic Function is defined thus: Analytic expressions which involve x, y only in the combination $x+iy$ were called by Cauchy monogenic functions.[0]

Holomorphic Function is defined thus: A function which is monogenic, uniform and continuous over any part of the z-plane is called holomorphic over that part of the plane.[1]

Examples.

(1). The algebraic expression $\frac{1}{1-z}$ defines a function $f(z)$ for values of z over the entire complex plane except at the point $z=1$.

(2). The values of a function of a complex variable may be defined for a certain region by means of a power series, as

$$f(z) = \cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} + \ldots\ldots$$

since $z=r(\cos\theta+i\sin\theta)$ the series $f(z)$ may be written

$$[1 - \frac{r^2}{2!}\cos 2\theta + \frac{r^4}{4!}\cos 4\theta \ldots\ldots] - i[\frac{r^2}{2!}\sin 2\theta - \frac{r^4}{4!}\sin 4\theta \ldots]$$

The series $f(z)$ converges for all finite values of z and therefore the function is defined over the entire complex plane except at $|z| = \infty$.

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* *Lectures on Theory of Functions of a Complex Variable*, Townsend, p. 50.

[0] *A Treatise on the Theory of Analytic Functions*, Harkness and Morley, p. 18.

[1] *Theory of Functions*, Forsyth, p. 15.

(3) A function of a complex variable may be defined by a definite integral, as $\Gamma(1+z) = \int_0^\infty e^{-t} t^z dt$ (3)

Integrating, we get a function of z for the right hand member of the equation leading to the functional equation*

$$\Gamma(1+z) = \Gamma(z).$$

(4) A definite integral along a given path will define a function, as

$$\int_0^z z^2 dz, \quad \text{or} \quad \int_0^{4+6i} \cos(zt)dt.$$

If the function is analytic the value of the integral will be the same for two different paths unless the loop they make encloses a singular point, or unless the function is multiform, when they may differ; while if it is not analytic the values may differ. The integral around any closed curve will be zero if the function is analytic and the curve does not surround a singular point.

8. Classes of functions may be defined by differential equations. The following are the Cauchy-Riemann differential equations,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \qquad (1)$$

which must hold for all points in the region of the functions, and they define a class of functions called analytic functions. If the differential equations given are not reducible to equations (1) the class of functions defined is not analytic. If we construct the second partial derivatives of $f(z)$ from the Cauchy-Riemann differential equations we get Laplace's differential equation.

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad (2)$$

This equation together with the equation

$$v = \int_0^z \left(-\frac{\partial u}{\partial y}dx + \frac{\partial u}{\partial x}dy\right) \qquad (3)$$

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* *Introduction to Analytic Functions*, Harkness and Morley, p. 20.

also defines an analytic function. If

$$\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial y^2} = m \neq 0$$

one of the necessary conditions that a function be analytic is not fulfilled and therefore the equation defines some class of functions other than analytic functions.

9. The problem of this thesis is to consider methods of finding expressions which hold over the various parts of the range of z for which the analytic function exists, that give the functional values for these parts. To consummate the analytic statement of the function values for all values of the variable z it is necessary to find analytic expressions which give the function values outside of the region which is the range of an initial expression, and which in the outer regions have the same analytic properties as the first expression has in its region, and further the various expressions must give the same numerical values where the various regions overlap. Each such expression is called a continuation of the first and each will be called an element of a function whose values are given by the totality of all the continuations possible from a given element.

10. The ordinary power series in a complex variable z is an example of an expression which is analytic and defines the values of an analytic function within its region of convergence. Let the power series be

$$P(z) = \alpha_0 + \alpha_1 z + \alpha_2 z^2 + \ldots\ldots \qquad (1)$$

where the α's are constants. If $P(z)$ converges when $z=z_0$ it converges absolutely when $|z|<|z_0|$. If $P(z)$ diverges when $z=z_1$ it diverges when $|z|>|z_1|$. Therefore there is in general some value R for which $|z|<R$ the series is absolutely convergent and for $|z|>R$ the series is divergent.

$$f(z) = \sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} + \ldots.. \qquad (2)$$

converges for $|z|<\infty$ so we see that there are series which converge over the entire finite part of the plane.

$$f(z) = z + 2!z^2 + 3!z^3 + \ldots. + n!z^n + \ldots. \qquad (3)$$

diverges for $|z|>0$ so R in this case has the value 0. If inside a circle of radius R the series $P(z)$ converges, and without that circle it diverges, this circle will be denoted by (R) and is called the circle of convergence. The region $[R]$ enclosed within this circle is called the domain of the series. This domain supplemented by the points on the circle at which $P(z)$ converges is called the region of convergence of $P(z)$ and forms the region of definition of the power-series mentioned above.

The following is Cauchy's theorem for the expansion of analytic functions in a series of integral positive powers of the variable, when the function values have already been defined in some other way:

*When a function is holomorphic over the area of a circle of center a, it can be expanded as a series of positive integral powers of z-a, converging for all points within the circle.**

Fig. 1.

Let z be any point within the circle; describe a concentric circle of radius r such that $|z-a| = \rho < r < R$, where R is the radius of the given circle. If t denotes a current point on the circumference of the new circle we have

$$f(t) = \frac{1}{2\pi i}\int\frac{f(t)dt}{t-z} = \frac{1}{2\pi i}\int\frac{f(t)}{t-a}\cdot\frac{dt}{1-\frac{z-a}{t-a}} \qquad (2)$$

the integral extending along the whole circumference of the circle of ra-

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* *Theory of Functions*, Forsyth, pp.42ff.

dius r. Now by dividing out we have

$$\frac{1}{1-\frac{z-a}{t-a}} = 1 + \frac{z-a}{t-a} + \left(\frac{z-a}{t-a}\right)^2 + \dots + \left(\frac{z-a}{t-a}\right)^n + \frac{\left(\frac{z-a}{t-a}\right)^{n+1}}{1-\frac{z-a}{t-a}} \qquad (3)$$

so that we have

$$f(z) = \frac{1}{2\pi i}\int\frac{f(t)dt}{t-a} + \frac{z-a}{2\pi i}\int\frac{f(t)dt}{(t-a)^2} + \dots + \frac{(z-a)^n}{2\pi i}\int\frac{f(t)dt}{(t-a)^{n+1}} + \frac{1}{2\pi i}\int\frac{f(t)}{t-z}\left(\frac{z-a}{t-a}\right)^{n+1}dt \qquad (4)$$

Now $f(t)$ is holomorphic over the whole area of the circle; hence if t be not actually on the boundary of the region, a condition secured by the hypothesis $r<R$, we have

$$f^{[s]}(a) = \frac{s!}{2\pi i}\int\frac{f(t)dt}{(t-a)^{s+1}} \qquad (5)$$

and therefore

$$f(z) = f(a) + (z-a)f'(a) + \dots + \frac{(z-a)^n}{n!}f^n(a) + \frac{(z-a)^{n+1}}{2\pi i}\int\frac{f(t)dt}{(t-z)}\frac{dt}{(t-a)^{n+1}} \qquad (6)$$

Let the last term be denoted by L. Since $|z-a|=\rho$ and $|t-a|=r$, it is at once evident that $|t-z|\geqq r-\rho$. Let M be the greatest value of $|f(t)|$ for points along the circle of radius r; then M must be finite, owing to the initial hypothesis relating to $f(z)$. Taking $t-a = re^{i\theta}$ so that $dt = i(t-a)d\theta$, we have

$$|L| = \frac{\rho^{n+1}}{2\pi}\left|\int_0^{2\pi}\frac{f(t)}{t-z}\cdot\frac{d\theta}{(t-a)^n}\right| < \frac{\rho^{n+1}}{2\pi}\cdot\frac{1}{r^n(r-\rho)}\int_0^{2\pi}|f(t)|d\theta \leqq \frac{\rho^{n+1}}{r^n(r-\rho)}M$$

$$= \left(\frac{\rho}{r}\right)^{n+1}M\left(1-\frac{\rho}{r}\right)^{-1} \qquad (7)$$

Now r was chosen to be greater than ρ; hence as n becomes infinitely large, we have $\left(\frac{\rho}{r}\right)^{n+1}$ approach 0. Also $M(1-\frac{\rho}{r})^{-1}$ is finite. Hence as n increases indefinitely, the limit of $|L|$, necessarily not negative, is zero and therefore in the same case L tends towards zero. It thus appears that when n is made to increase without limit, the difference between the quantity $f(z)$ and the first $n+1$ terms of the series approaches zero;

hence the series is a converging series having $f(z)$ as the limit of the sum, so that

$$f(z) = f(a) + (z-a)f'(a) + \frac{(z-a)^2}{2!}f''(a) + \ldots + \frac{(z-a)^n}{n!}f^{[n]}(a) + \ldots \quad (8)$$

which proves the proposition, and the form of Taylor's expansion for complex variables. The above converging power series is denoted by $P(z-a)$ or $P(z|a)$.

11. The following facts relating to the above proof are important.

(1) In order that $\frac{1}{t-z}$ may be expanded into the required form, the point z must be taken actually within the area of the circle of radius R; and therefore the convergence of the series $P(z-a)$ is not established for points on the circumference.

(2) The coefficients of the powers of $z-a$ in the series, multiplied by factorial n, are the values of the function and its derivatives at the center of the circle; and the existence of the derivatives is assured by the holomorphic character of the function for all points within the region. It therefore follows that if a function be holomorphic within a region bounded by a circle of center a, its expansion in a series of ascending powers of $z-a$, which converges for all points within the circle, depends only upon the values of the function and its derivatives at the center.

But instead of having the values of the function and all its derivatives at the center of the circle, it will suffice to have the values of the holomorphic function itself over any region around a or along any line thru a. The values of the derivatives at a can be found in either case; for $f'(b)$ is the limit of $\frac{f(b+\Delta b)-f(b)}{\Delta b}$, so that the value of the first derivative can be found for any point in the region or on the line,

as the case may be; and so for all the derivatives in succession.

The following is Laurent's theorem:

*A function which is holomorphic in a part of the plane bounded by two concentric circles and having finite radii, can be expanded in a series of integral powers, positive and negative, of z-a; and the series converges in the part of the plane between the circles.**

The proof of this theorem is similar to that of Cauchy's just given and the expression representing this function is $P_1(z-a) + P_2(\frac{1}{z-a})$, the power series P_1 converging within the outer circle and the power series P_2 converging without the inner circle; their sum converges for the ring-space between the circles.

12. Definitions of classes of points which arise in the consideration of uniform functions. A point of the plane may be such that a function of the variable has a determinate finite value there, always independent of the path by which the variable reaches a; the point a is called an ordinary point or a regular point of the function.

The point a in the plane may be such that a function $f(z)$ of the variable has a determinate infinite value there, always independent of the path by which the variable reaches a, the function behaving regularly for points in the vicinity of a; then $\frac{1}{f(z)}$ has a determinate zero value there, so that a is an ordinary point of $\frac{1}{f(z)}$. The point a is called a pole or a non-essential singularity of the function.

A point a may be such that $f(z)$ has not a determinate value there, either finite or infinite, though the function is definite in value at all points in the immediate vicinity of a other than a itself. Such a

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* *Theory of Functions*, Forsyth, p. 47.

point is called an essential singularity of the function. These cases given are not by any means exhaustive, but sufficient for the purpose of this thesis.

Let $f(z)$ denote the function represented by a series of powers $P_1(z-a)$, then the circle of convergence bounds what may be called the domain of the ordinary point a of the function. At this point the series takes a definite finite value, is continuous, and has derivatives of all orders. Any derivative $P^{[n]}(z-a)$ is holomorphic inside the circle of convergence of the given series.

The circle of convergence may be infinitely large. Thus the exponential function

$$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \ldots\ldots + \frac{z^n}{n!} + \ldots\ldots$$

expressed in the given power series is valid over the entire plane except at the point $z=\infty$.

CHAPTER II.

METHODS OF ANALYTIC CONTINUATION BY CONSTRUCTING AN EXPRESSION IN POLYNOMIALS VALID OVER A WIDER FIELD.

13. *The Weierstrass Continuation.* When the region of the initial power-series of a function $f(z)$ is finite, but $f(z)$ is holomorphic over a region which extends beyond the domain of a, the center of convergence of the power series, although the circumference bounding that domain is the largest with center a that can be drawn within the region, the region evidently cannot extend beyond the domain of a in all directions.

Take an ordinary point b in the domain of a. The value at b of the function $f(z)$ is given by the series $P(b-a)$, and the values at b of all its derivatives are given by the derived series. All these series converge within the domain of a and they are therefore finite at b and their expressions involve the values at a of the function and its derivatives.

Let the domain of b be considered. The domain of b may be included in that of a, and then its bounding circle will touch the bounding circle of the domain of a internally. If the domain of b be not entirely included in that of a, part of it will lie outside of the domain of a; but it cannot include the whole of the domain of a unless its bounding circumference touches that of a externally, for otherwise it would extend beyond a in all directions, a result inconsistent with the hypothesis as to the domain of a. Hence there must be points excluded from the domain of a which are also excluded from the domain of b. For all points z in the domain of b, the function can be represented by a series, say

$P_2(z-b)$, the coefficients of which are the values at b of the function and its derivatives divided by $n!$. Since these values are partially dependent upon the corresponding values at a, the series representing the function may be denoted by $P_2(z-b,a)$.

When $f(z)$ is a uniform function, then at point z in the domain of b lying also in the domain of a, the two series $P_1(z-a)$ and $P_2(z-a,b)$ must furnish the same numerical value for the function $f(z)$ and may give the same value in the case of a multiform function; and therefore no new value is derived from the new series P_2 which cannot be derived from the old series P_1. For all such points the new series is of no advantage, and hence if the domain of b be included in that of a, the construction of the series $P_2(z-b,a)$ is superfluous. Hence in choosing the ordinary point b in the domain of a we choose a point, if possible, that will not have all its domain included in that of a.

At a point z which does not lie within the domain of a but which does lie in the enlarged region, the series $P_2(z-b,a)$ gives a value for $f(z)$ which cannot be given by $P_1(z-a)$. The new series $P_2=P_2(z-b,a)$ then gives an additional expression for the function; it is called an analytic continuation of the first series which represents the function in the domain of a. Also the derivatives of P_2 give the values of the derivatives of $f(z)$ for points in the domain of b.

It thus appears that, if the whole of the domain of b be not included in that of a, the analytic expressions for the values of the function can, by a series which is valid over the whole of a second domain be continued into that part of the second domain excluded from the domain of a.

Now take a point c within the region occupied by the combined domain of a and b, and construct the domain of c. In the new domain the function values can be expressed by a new series, say $P_3=P_3(z-c)$ or, since the coefficients (being the values at c of the function and its derivatives divided by the factorials) involve the values of the function and its derivatives at a and possibly also the values at b, the series expressing the function may be denoted by $P_3(z-c,a,b)$. Unless the domain of c includes points, which are not included in the combined domains of a and b, the series P_3 does not give any value of the function which cannot be given by P_1 or P_2; we therefore choose c if possible so that its domain will include points not included in the earlier domains. At such points z in the domain of c as do not belong to the combined domains of a and b, the series $P_3(z-c,a,b)$ gives a value for $f(z)$ which cannot be derived from P_1 or P_2; and thus the new series is an analytic continuation of the earlier series.

Proceeding in this manner by taking successive points and constructing their domains, we can cover more or less of the plane by a series of overlapping circles so as to include all points inside some region in which the function preserves its holomorphic character; this region is called a region of continuity of the function. An analytic function cannot have two or more isolated regions of continuity. With each domain so constructed as to include some portion of the region of continuity is associated a power series, which is an analytic continuation of an earlier series, and, as such, gives values of the function not calculable from these earlier series; all the associated series are ultimately calculated from the first. The region of continuity so attained may be bounded

by a finite closed curve over which no continuation is possible. Such a boundary is called a natural boundary. An analytic function can not have more than one region, each with a natural boundary.

Each of the series is called an element of the function. The aggregate of all the distinct elements is called a monogenic analytic function; it is a complete analytical expression of the function in its region of continuity. This is the Weierstrassian definition* of a monogenic analytic function.

Let z be any point in the region of continuity, not necessarily in the circle of convergence of the initial element of the function; a value of the function at z can be obtained thru the continuations of that as an initial element. Hence the following theorem:

If we know the value of the analytic function $f(z)$ and of all of its successive derivatives at a fixed point a of the region of continuity A, we may then deduce the value of this function at any other point b of the region A.°

When the analytic function is uniform, the same value at z for the function is obtained whatever be the set of domains. If there be two or more sets of elements, differently obtained which give at z different values for the function, the analytic function is multiform; but not every new set of elements leads to a change in the value at z of a multiform function, and a multiform analytic function has a uniform character within any region of the plane that admits only equivalent sets of elements.

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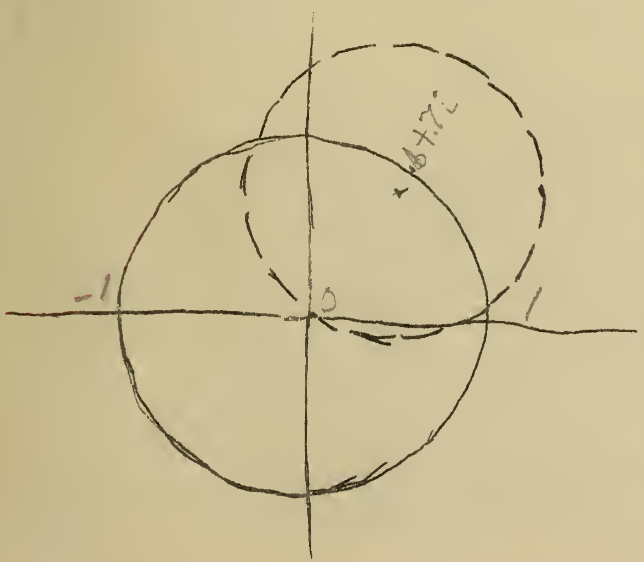
* *Theory of Functions*, Forsyth, p. 56.

° *Cours d'Analyse Mathématique*, E. Goursat, II p. 287.

The whole process is reversible for any analytic function . Any one of the analytic continuations of a uniform analytic function, represented by a power series can be deduced from any other; and therefore the expression of such a function in its region of existence is potentially given by one element, for all the distinct elements can be deduced from any one element.

The singularities of a function limit its region of continuity; for each of the separate domains is, from its construction, limited by its nearest singularity, and the combined aggregate of domains constitutes the region of existence. The complete boundary of a region of existence is a locus on which lie all the singularities of the function. It also follows that in order to know a uniform analytic function it is only necessary to know some one element of the function. The chief source of interest in the development of the theory of analytic functions is indeed due to the fact that from a single expression holding only over a limited range may be found expressions for every other portion of the domain of existence.

The following is an example illustrating the Weierstrass method of continuation. Start with the series:



$$P(z_0)=1+z_0+z_0^2+ \dots\dots + z_0^n+ \dots$$

then

$$P'(z_0)=1+2z_0+ \dots\dots + nz_0^{n-1}+\dots$$

and

$$P''(z_0)=2+ \dots\dots + n(n-1)z_0^{n-2} + \dots$$

$$\dots\dots\dots\dots\dots\dots\dots\dots$$

We know that $P(z_0)$ converges within a circle of radius 1 about the ori-

gin as center.

Let us continue the expressions for the values by a Taylor's series about the point $z_0=.6+.7i$. We know that

$$P(z-z_0)=P(z_0)+(z-z_0)\frac{P'(z_0)}{1!}+(z-z_0)^2\frac{P''(z_0)}{2!}+\ldots\ldots \tag{2}$$

But $P(z_0) = 1.23(\cos 60^\circ + i\sin 60^\circ)$,

$$\begin{aligned} P'(z_0) &= 1.54(\cos 120^\circ + i\sin 120^\circ),\\ P''(z_0) &= 3.85(\cos 180^\circ + i\sin 180^\circ),\\ P'''(z_0) &= 14.01(\cos 240^\circ + i\sin 240^\circ). \end{aligned} \tag{3}$$

The above calculations are only approximations. Substituting for z_0

$$\begin{aligned} P(z-z_0) = {} & 1.23(\cos 60^\circ + i\sin 60^\circ) + (z-.6-.7i)(1.54)(\cos 120^\circ + \\ & i\sin 120^\circ) + (z-.6-.7i)^2\frac{(3.85)}{2\cdot 1}(\cos 180^\circ + i\sin 180^\circ) + \\ & (z-.6-.7i)^3\frac{14.01}{1\cdot 2\cdot 3}(\cos 240^\circ + i\sin 240^\circ) + \ldots\ldots \end{aligned} \tag{4}$$

which gives

$$\begin{aligned} P(z\,|\,.6+.7i) = {} & .615 + 1.069i + (z-.6-.7i)(-.77+1.43i) + \\ & (z-.6-.7i)^2(-1.92) + (z-.6-.7i)^3(-1.165-2i) + \ldots. \end{aligned} \tag{5}$$

$z'=z-z_0$ so for $z=.9+.1i$, $z'=.3-.6i$, which is a point in the region common to $P(z)$ and $P(z|z_0)$, the numerical value of the limit of each series being $5+5i$.

The radius of convergence of $P(z-z_0)$ is $|z|=\sqrt{.4^2+.9^2} = \sqrt{.65} = .8+$ so that $P(z-z_0)$ converges within a circle about $z_0=(.6+.7i)$ of radius $\leqq .8$. Therefore $P(z-z_0)$ is an analytic continuation of $P(z)$. It is readily seen that the Weierstrassian method of analytic continuation is very impractical for use in computation because the numerical limit of a series must be calculated to get each of the coefficients in a new series and the number of series that are necessary becomes very large.

14. *The Mittag-Leffler Continuation involving a star-shaped region.*

If K is a continuum enclosing a point a, bounded by a single curve which does not intersect itself, and such that the branch of the function $f(z)$,* made up by $P(z|a)$ and its analytic continuation interior to K, remains uniform and regular, let this branch be designated by $fK(z)$.

The problem is to find an analytic representation of the branch $fK(z)$ for the region of greatest possible extension.

From the definition of the analytic function $f(z)$ and of the branch $fK(z)$, there immediately results a manner of representing the branch $fK(z)$ analytically. To do this it is only necessary to work out a denumerable number of analytic continuations of $P(z|a)$ for example

$$P_\nu(z|a_\nu) = \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{d^n fK(z)}{dz^n}\right)(z-a_\nu)^n, \quad \begin{array}{l} \nu=0,1,2,\ldots\,; \\ a_0=a; \\ P_0(z|a_0)=P(z|a). \end{array}$$

The series $P_\nu(z|a_\nu)$ are formed by means of the derivatives $\left(\frac{d^n fK(z)}{dz^n}\right)_{z=a_\nu}$ $\left(\begin{array}{l} n=0,1,2,\ldots \\ \nu=0,1,2,\ldots \end{array}\right)$ divided by the factorials, and these coefficients may again be calculated from the quantities $f^{(n)}(a)$, $(n=0,1,2,\ldots;\ f^{(0)}(a)=f(a))$.

But the carrying out of this calculation requires the knowledge of the radius of convergence of each series $P_\nu(z|a_\nu)$.** We have Cauchy's theorem where we have the radius of convergence expressed by the reciprocal of the upper limit of the positive quantities

$$\left| \sqrt[n]{\frac{1}{n!}\left(\frac{d^n fK(z)}{dz^n}\right)_{z=a}} \right| \quad (n=0,1,2,\ldots).$$

At first sight it seems that Cauchy's theory, which is built up on principles quite different from that of Weierstrass, possesses a great advantage over the other, when it comes to the analytic representation of $fK(z)$. Such a representation is given by the formula

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* $f(z)$ may be a multiform function.

** *Acta Mathematica*, v. 28, p. 45.

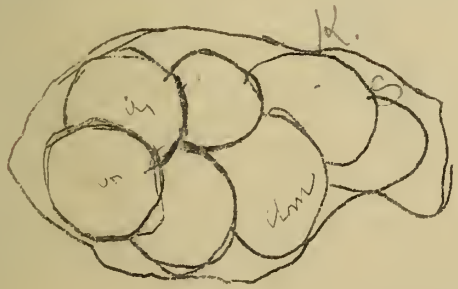
$$fK(z) = \frac{1}{2\pi i}\int_S \frac{fK(t)}{t-z}dt,$$

where the integral is taken along a closed contour S situated in the interior of K and approaching the boundary of K as close as we please.

This means the taking of the integral of the series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a_\nu)}{n!}(z-a_\nu)^n \quad (\nu=0,1,2,\dots\, m) \qquad (1)$$

around the contour S where the number of values we allow ν to take approaches m in proportion as the contour S approaches the boundary K. In doing this we can sum all of the constant terms of (1) and take the integral of this sum around the contour S, and then sum all of the coefficients of z and take the integral of this sum times z around the contour S and continuing in this way, by summing all of the results, we finally get the value of the above integral. The contour S is made up of a number of circular arcs, each arc being a part of the circle of convergence of the ν^{th} element of the analytic continuation of $P(z|a)$, and since the integral of each of the above sums must be taken around each arc separately, and one must know each element of the analytic continuation, we can see that it is a task involving no small amount of labor. However



this establishes the existence of a single representation of f for all the region K.

According to the same definition of an integral, it is evident that the above integral may be replaced by an infinite sum of rational functions of z where the coefficients are expressed by a denumerable number of special values of z (which are given by a_ν, as ν takes the different possible values) and by the corresponding values of $fK(z)$. Thus by the aid of the

knowledge of the quantities $f(a)$, $f'(a)$, $f''(a)$, ... $f^{(n)}(a)$, ... alone, a formula may be constructed representing the branch $fK(z)$ in its entirety.

Let us build up a domain according to the Mittag-Leffler method. Take the values at a of $f(a)$, $f'(a)$, $f''(a)$, ... and form the series

$$P(z|a) = \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(a)(z-a)^n.$$

($f(a)$ is a MacLaurin's series in a). Effect the analytic continuations of $P(z|a)$ along a vector issuing from the point a. It may be possible to continue $P(z|a)$ along the vector to infinity. Otherwise we come to some point a_i on the vector along which the analytic continuation of $P(z|a)$ is impossible. The part of the vector reaching from a_i to infinity is excluded from the domain of expression of the function. Rotating the vector around a we repeat the above process for every position. The domain of expression of the function is called by some a star as it takes in the entire z-plane with the exception of such lines as are excluded by the above process, as for example the line extending from a_i to infinity.

This star is given in a unique manner when the quantities $f(a)$, $f'(a)$, $f''(a)$, ... are fixed and we call it the star pertaining to the quantities. We designate it in general by A. The branch is called $fA(z)$.

Instead of using straight lines as vectors we could have used any convenient curves.

The following is Mittag-Leffler's theorem, which givesan analytic representation for the function valid over the entire star.*

The branch $fA(z)$ may always be represented by the series $\sum_{n=0}^{\infty} G_n(z)$ where $G_n(z)$ designates rational integral functions of z.

$G_n(z) = \sum_{\nu=0} C_\nu^{(n)} f^{\nu}(a)(z-a)^{\nu}$, where the coefficients C_ν are given a pri-

*Acta Mathematica , v.23, p.49.

ori independently of the choice of a and of $f^{(\nu)}(a)$, $(\nu=0,1,2,\ldots\ldots)$. *This series $G_n(z)$ is convergent for each point of the star A and uniformly convergent for each domain interior to A.*

He proves on page 62 also the final theorem

If A is the star belonging to f(a), f'(a), f"(a) ..... and fA(z) the functional element corresponding to these coefficients, then fA(z) can always be represented by a series $\sum_{n=0}^{\infty} G_n(z)$ *where the terms $G_n(z)$ are polynomials of the form* $G_n(z) = \sum_{\nu=0}^{m} c_\nu^{(n)} f^{(\nu)}(a)(z-a)^\nu$, *each coefficient $c_\nu^{(n)}$ being a rational determinate number which depends only on ν and n. The series* $\sum_{n=0}^{\infty} G_n(z)$ *is convergent for each value of z inside A and is uniformly convergent for every domain interior to A.*

Borel has proved this theorem by using the expression $\frac{1}{1-z}$. Painlevé* has found a general expansion for this expression.

$$\text{Let } \zeta = g(Z) = \left(\frac{2^N+1+Z(2^N-1)}{2^N+1-Z(2^N-1)}\right)^{\frac{1}{N}} - 1.$$

This function converts a unit circle in the Z-plane into a sector in the ζ-plane with its vertex at -1 and rays at angles $\pm\frac{\pi}{2N}$ as to the axis of reals. $Z=0$ corresponds to $\zeta=0$, and $Z=1$ to $\zeta=1$.

If Z describes the unit circle in its plane $z=\zeta^2$ will describe a closed curve C_N through $z=1$ and surrounding the origin. If $N\to\omega$, $C_N\to$ the segment 0 to 1. If $z=\frac{a}{\zeta^2}$, z describes a closed curve C_N^a, which for $N\to\omega$ tends towards the segment 1 to $+\omega$.

Expand by MacLaurin's theorem

$$\varphi(Z) = \frac{1}{1-z\zeta^2} = A_0(z) + A_1(z)Z + A_2(z)Z^2 + \ldots\ldots$$

the coefficients A_{2q}, A_{2q+1} being polynomials in z of degree q. Let $Z=1$ and $B_q(z) = A_{2q}(z) + A_{(2q+1)}(z)$, then $\sum_0^\infty B_q(z) \equiv \sum_0^\infty(\lambda_{0,q}(N) + \lambda_{1,q}(N)Z + \ldots + \lambda_{q,q}(N)(z)^\nu$. (1)

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* C.R. 129, pp. 28-29.

The λ's are rational functions of N with integral coefficients.

(1) $\to \frac{1}{1-z}$ in C'_N, and from this may be determined an expansion in polynomials thus: Let $Q_n \equiv S_n^N$ be the expression found by writing the first n terms of the series in $B_q(z)$, in which we set $N = \log(n+2)$; and let $\pi_0 = 1$, $\pi_n = Q_{n+1} - Q_n$, then

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} \pi_n(z).$$

This gives

$$Q_1 = 1.$$

$$Q_2 = \left(\frac{4}{N^2}m^2 + \frac{8}{N^3}m^3\right)z \text{ and } m = \frac{2^{\log 4}-1}{2^{\log 4}+1}, \quad N = \log 4$$

$$Q_3 = \left[\frac{4(7+2N^2)}{3N^4}m^4 + \frac{8(1+6N-N^2)}{3N^5}m^5\right]z + \left[\frac{16m^4}{N^2} + \frac{64m^5}{N^5}\right]z^2 \text{ and } N = \log 5, \quad m = \frac{2^{\log 5}-1}{2^{\log 5}+1}.$$

Hence

$$\pi_0 = 1.$$

$$\pi_1 = \left[4\left[\frac{\frac{2^{\log 4}-1}{2^{\log 4}+1}}{\log 4}\right]^2 + 8\left[\frac{\frac{2^{\log 4}-1}{2^{\log 4}+1}}{\log 4}\right]^3\right]z - 1.$$

$$\pi_2 = z\left[\frac{4(7+\log 5^2)}{3\log 5^4}\left(\frac{2^{\log 5}-1}{2^{\log 5}+1}\right)^4 + \frac{8(1+6\log 5-\log 5^2)}{3\log 5^5}\left(\frac{2^{\log 5}-1}{2^{\log 5}+1}\right)^5 - 4\left[\frac{\frac{2^{\log 4}-1}{2^{\log 4}+1}}{\log 4}\right]^2 - 8\left[\frac{\frac{2^{\log 4}-1}{2^{\log 4}+1}}{\log 4}\right]^3\right] + z^2\left[\frac{16\left(\frac{2^{\log 5}-1}{2^{\log 5}+1}\right)^4}{\log 5^2} + 64\left[\frac{\frac{2^{\log 5}-1}{2^{\log 5}+1}}{\log 5}\right]^5\right]$$

In a note in Leçons sur les fonctions de Variables réelles, et les développements en séries de polynomes, by Borel, p. 120, Painlevé gives a simpler form and one indeed applicable to any function. It is as follows: Let $K_l(n) = n(n+1)\ldots\ldots(n+l-1)$, and let it be understood that when n is of the form $\alpha f'$ that

$$(\alpha f')^{(n)} = \alpha^n f^{(n)}.$$

Then if $\alpha = \frac{2}{\log n}$, $\beta = 1 - \frac{1}{\sqrt{n}}$ and we form

$$Q_0 = f(0)$$

$$Q_n = f(0) + \sum_1^n \frac{3^l}{l!} K_l(\alpha z f'(0))$$

then $\pi_0 = 1$, $\pi_n = Q_n - Q_{n-1}$, the series $\pi_0 + \pi_1 + \pi_2 + \ldots$ represents $f(z)$ in all its star. For example if $f(z) = \frac{1}{1-z}$, so that $f(0)=1$, $f'(0)=1$, $f''(0)=2!$ ..... $f^{(n)}(0)= n!$ then

$$K_1(\alpha z f_0') = \alpha z$$

$$K_2(\alpha z f_0') = 2!\alpha^2 z^2 + \alpha z$$

$$K_3(\alpha z f_0') = 3!\alpha^3 z^3 + 3\cdot 2!\alpha^2 z^2 + 2\alpha z$$

$$K_4(\alpha z f_0') = 4!\alpha^4 z^4 + 6\cdot 3!\alpha^3 z^3 + 11\cdot 2!\alpha^2 z^2 + 6\alpha z$$

....

Hence

$$Q_0 = 1.$$

$$Q_1 = 1 + 0 = 1$$

$$Q_2 = 1 + \frac{(1-\frac{1}{\sqrt{2}})}{1}\cdot\frac{2}{\log 2}z + \frac{(1-\frac{1}{\sqrt{2}})^2}{2}[2!\frac{2^2}{\log 2^2}z^2 + \frac{2}{\log 2}z]$$

$$Q_3 = 1 + \frac{(1-\frac{1}{\sqrt{3}})}{1}(\frac{2}{\log 3}z) + \frac{(1-\frac{1}{\sqrt{3}})^2}{2}(2!\frac{2^2}{\log 3^2}z^2 + \frac{2}{\log 3}z)$$
$$+ \frac{(1-\frac{1}{\sqrt{3}})^3}{3}[3!\frac{2^3}{\log 3^3}z^3 + 3\cdot 2!\frac{2^2}{\log 3^2}z^2 + \frac{2}{\log 3}z].$$

Hence π_0, π_1, π_2, ...

$$\pi_0 = 1,$$

$$\pi_1 = 0,$$

$$\pi_2 = z\frac{2}{\log 2}(2-\frac{1}{\sqrt{2}}) + z^2\frac{2^2}{\log 2^2}(1-\frac{1}{\sqrt{2}})^2$$

$$\pi_3 = z[\frac{4}{\log 3}\cdot\frac{(1-\frac{1}{\sqrt{3}})^3}{3} + \frac{(1-\frac{1}{\sqrt{3}})^2}{\log 3} - \frac{(1-\frac{1}{\sqrt{2}})^2}{\log 2} - \frac{2(1-\frac{1}{\sqrt{2}})}{\log 2}]$$
$$+ z^2[\frac{2^2}{\log 3^2}(1-\frac{1}{\sqrt{3}})^3 + \frac{2^2}{\log 3^2}(1-\frac{1}{\sqrt{3}})^2$$
$$-\frac{2^2}{\log 2^2}(1-\frac{1}{\sqrt{2}})^2] + z^2\frac{2^3}{\log 3^3}(1-\frac{1}{\sqrt{3}})^3].$$

In general if $F(z)$ is a branch of an analytic function, holomorphic at 0, α its star, a any one of the singular points of the branch $F(z)$,

a is a summit of the star, and let α_N be the area inside all the curves C_N^q as $N \to \infty$, $\alpha_N \to \alpha$.

In the above expression (1) replace z^j by $z^j \frac{f^{(j)}(0)}{j!}$ and $P_q(z)$ will be what $B_q(z)$ becomes, then

$$F(z) = \sum_{q=0}^{\infty} P_q(Z) = \sum_{q=0}^{\infty} [F(0)\lambda_{0,q}^N + \frac{F'(0)}{1!}\lambda_{1,q}^N Z + \dots + \lambda_{q,q}^N \frac{F^q(0)}{q!} z^q].$$

Out of this series we build the polynomial expansion in the same way as for $\frac{1}{1-z}$.

CHAPTER III.

METHODS OF ANALYTIC CONTINUATION BY CONFORMAL GEOMETRIC TRANSFORMATIONS.

15. *The Lindelöf Method of Analytic Continuation.* The method of Lindelöf rests essentially upon the employment of conformal representation. Let D be a simply connected domain in the z-plane and W an arbitrary circle situated in the w-plane, for example a circle with the origin as center and unit radius. A fundamental proposition* due to Riemann shows that there is an analytic uniform function within D , $g(z)$, such that the transformation $w=g(z)$ establishes a one to one conformal correspondence between the domains D and W. Inverting the equation $w=g(z)$ we get $z=\varphi(w)$ holomorphic in the circle W on account of the properties of conformal representation.

Having done this let $f(z)$ be a holomorphic function existing within D; its development in a power series according to the powers of $(z-z_0)$ [z_0 denotes an arbitrary point interior to D] converges within a circle; thus the function is represented by a Taylor's series within a part at least of the domain in which it exists, say D'. But let us make the substitution $z=\varphi(w)$. The function $f(z)$ becomes a function $\psi(w)$, holomorphic within a region W' which is what the region D' becomes by $z= \varphi(w)$. This function $\psi(w)$ is expressible by a power series of the form

$$P(w) = \alpha_0 + \alpha_1 w + \alpha_2 w^2 + \ldots\ldots + \alpha_n w^n + \ldots.$$

valid in a circle W'' included in W'. This function $P(w)$ is holomorphic however not only in the domain corresponding to the circle of convergence of $P(z-z_0)$, but in all of the circle W according to Cauchy's theorem. It

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* Fouét, "Leçons sur les Fonctions Analytiques", vol. 2, pp. 98ff.

may be added that the transformations used by Lindelöf may serve not only to extend functions outside thecircle of convergence of their initial element, but also to find their singular points, and above all to replace these developments by developments more convergent*.

Example.- Let us consider the transformation $w=u+iv=\tan z$. To find out what the unit circle in the w-plane maps into in the z-plane we first find u and v by setting $u+iv=\tan(x+iy)$.

$$\tan(x+iy) = \frac{\tan x + \tan iy}{1 - \tan x \tan iy} = \frac{\tan x - i\tanh y}{1 + i\tan x\tanh y}.$$

Then rationalizing we have

$$\tan(x+iy) = \frac{\tan x - \tan x\tanh^2 y - i(\tan^2 x\tanh y + \tanh y)}{1 + \tan^2 x\tanh^2 y}.$$

$$u = \frac{\tan x(1 - \tanh^2 y)}{1 + \tan^2 x\tanh^2 y}$$

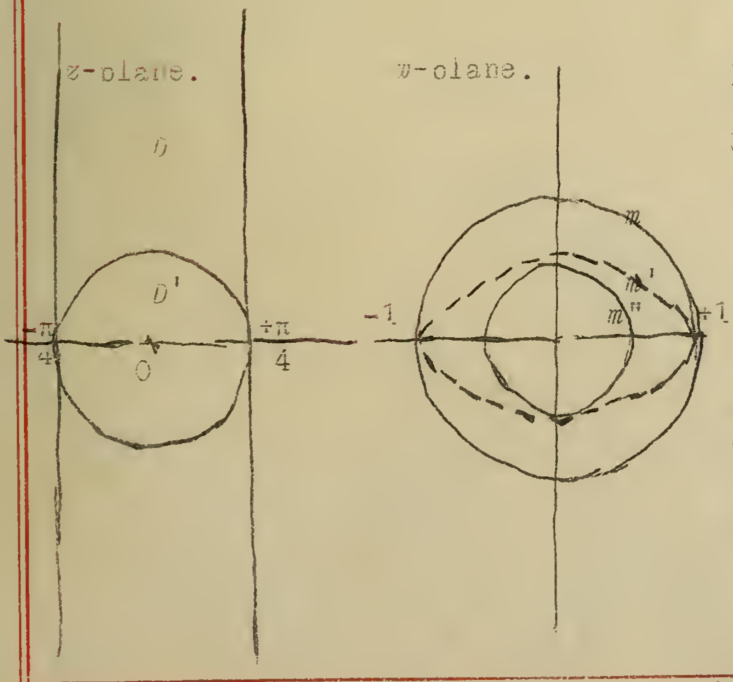
$$v = -\frac{\tanh y(\tan^2 x + 1)}{1 + \tan^2 x\tanh^2 y}$$

Then we have the equation

$$u^2 + v^2 = \left[\frac{\tan x(1 - \tanh^2 y)}{1 + \tan^2 x\tanh^2 y}\right]^2 + \left[\frac{\tanh y(\tan^2 x + 1)}{1 + \tan^2 x\tanh^2 y}\right]^2$$

$$= \frac{\tan^2 x(1 + \tan^2 x\tanh^2 y) + \tanh^2 y(1 + \tan^2 x\tanh^2 y)}{(1 + \tan^2 x\tanh^2 y)^2}$$

$$= \frac{\tan^2 x + \tanh^2 y}{1 + \tan^2 x\tanh^2 y} = 1$$



$$(\tan^2 x - 1)(1 - \tanh^2 y) = 0.$$

Hence $\tan x = \pm 1$, $\tanh y = \pm 1$.

So we see that the unit circle in the w-plane maps into the two lines $x=\frac{\pi}{4}$ and $x=-\frac{\pi}{4}$ in the z-plane and the region inside this circle maps into the region included be-

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*Fouét, "Leçons sur les Fonctions Analytiques", p. 100.

tween these two lines.

The region in the circle D' together with its bounding circumference maps into the region W' and its bounding curve. The points $\pm\frac{\pi}{4}$ map into the points ± 1 and the points $\pm\frac{\pi}{4}i$ map into the points $\pm .6557i$ (since $\tanh\frac{\pi}{4} = .6557$).

Consider now the function $f(z) = \dfrac{1}{\frac{\pi}{4}-z}$ which is known to be holomorphic in the region D; set $\frac{\pi}{4}=\alpha$, and let $f(z) = \frac{1}{\alpha} + \frac{z}{\alpha^2} + \frac{z^2}{\alpha^3} + \ldots.$ (1)

The series $f(z)$ converges within the circle D' with radius $\alpha=\frac{\pi}{4}$. We derive

$$\varphi(w) = \frac{1}{\alpha} + \frac{\tan^{-1}w}{\alpha^2} + \frac{(\tan^{-1}w)^2}{\alpha^3} + \ldots.. \qquad (2)$$

Series (2) is convergent in the region W' because the region D' maps into the region W'. But

$$P_1(w) = \frac{1}{\alpha} + \frac{w}{\alpha^2} + \frac{w^2}{\alpha^3} + \frac{(3-\alpha^2)w^3}{3\alpha^4} + \frac{(3-2\alpha^2)w^4}{3\alpha^5} + \ldots.. \qquad (3).$$

Series (3) converges° in the circle W'' whose radius is .6557. But $\varphi(w)$ is also holomorphic in W. Therefore $P_1(w)$ represents $g(w)$ in W , and passing back to z we have the series

$$Q(z) = \frac{1}{\alpha} + \frac{\tan z}{\alpha^2} + \frac{\tan^2 z}{\alpha^3} + \frac{3-\alpha^2}{3\alpha^4}\tan^3 z + \frac{3-2\alpha^2}{3\alpha^5}\tan^4 z + \ldots.. \qquad (4)$$

and $Q(z) \equiv f(z)$ in D because of the conformal representation of W upon D. The region D overlaps the region D'. So $f(z)$ which is convergent in D' is equal to $Q(z)$ in D' and $Q(z)\equiv f(z)$ in D and $f(z)$ is holomorphic in D. Therefore $Q(z)$ is an analytic continuation of $P(z)$ from D' into D.

16. *The Schwarz Continuation.* Let $f(z)$ be a function* holomorphic within a domain D of which the boundary C includes a segment of the real axis OX, we suppose that this function tends towards a value, well determined and real, $f(x)$ when the point z approaches by any path whatever interior to D, the point x on the real axis, and we further suppose that the values of the function along the segment of the real axis form a con-

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* Goursat, *Leçons sur les Fonctions Analytiques*, vol.2, p. 101.

° At least

tinuous set of values.

Represent by $\bar{z}$ the symmetrical point of z with respect to OX, and associate to each point $\bar{z}$ the value of the conjugate $\overline{f(\bar{z})}$ of $f(z)$. The set of points $\bar{z}$ defines a domain $\bar{D}$ symmetric to D, and the set of numbers $\overline{f(\bar{z})}$ defines as can be easily shown a function holomorphic within $\bar{D}$. The two holomorphic functions $f(z)$ and $\overline{f(\bar{z})}$, one on the first and the other on the second side of C, have the same value on the segment of the real axis; and the one expression is an analytic continuation[1] of the other expression and both belong to the function. Thus not only may the function $f(z)$ be continued within the image domain of D, but to obtain this continuation it is sufficient to consider the values which are the conjugates of the values which $f(z)$ takes in D and to associate them to the points $\bar{z}$ conjugates of the points z of D.

To generalize this* we replace the segment of the real axis by a straight segment arbitrarily drawn in the z-plane. Then every function holomorphic within a domain where the boundary contains the rectilinear segment and which takes continuous determinate real values on the segment as defined above is continuable within the image-domain of the primitive domain, and its continuation is obtained by associating to the points z' symmetrical to the points z of the first domain with respect to the segment, values of the function determined from the values at the symmetric points within the first domain. Schwarz has proved that this continuation by symmetry is an analytic continuation.

Painlevé has put into simple form° the necessary and sufficient conditions that an analytic function be continuable beyond a regular arc of

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* *Fouët, Leçons sur les Fonctions Analytiques*, vol. 2, p. 101.

° *Ibid*, vol. 2, p. 104.

[1] *Forsyth, Theory of Functions*, pp. 57-58.

an analytic curve. Let $x=x(t)$, $y=y(t)$ be the equation of this curve C at all points.

That the function $f(z)$ defined on the side c of C be continuable on the side $\bar{c}$, it is necessary and sufficient that $f(z)$ take on a portion of C a set of values $f_1(t)$ which is an analytic function of t.

Schwarz has given the theorem* in a still more simple form as follows:

That the function $f(z)$ be continuable on the side $\bar{c}$, it is necessary and sufficient that its real part $u(x,y)$ (or its imaginary part v) take on a portion of C a set of values $u_1(t)$, which is an analytic function of t.

Example- Let $w = f(z) = \int_0^z \frac{dz}{\sqrt{(1-z^2)(1-k^2z^2)}}$ where z is limited to the real axis and upper half of the z-plane. Then the values of w lie in a

rectangle. If now Schwarz symmetric continuation is applied to the function, we have the values of z continued into the lower half of the z-plane, while the w values lie inside of the reflection of the rectangle OABC in the real axis.

17. Schwarz has further worked out an expression which will transform the area inside of a polygon in the w-plane into the upper half of the z-plane, or into any circle. This method is of use in Lindelöf's process. In the transformation the boundary of the polygon becomes the axis of reals in the z-plane. If we let a, b, c, ..., l be the points on the axis of x corresponding to the angular points of the polygon, and let $\alpha\pi$, $\beta\pi$, $\gamma\pi$, ..., $\lambda\pi$ be the internal angles of the polygon at the respec-

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* *Fouét, Leçons sur les Fonctions Analytiques*, vol. 2, p. 105.

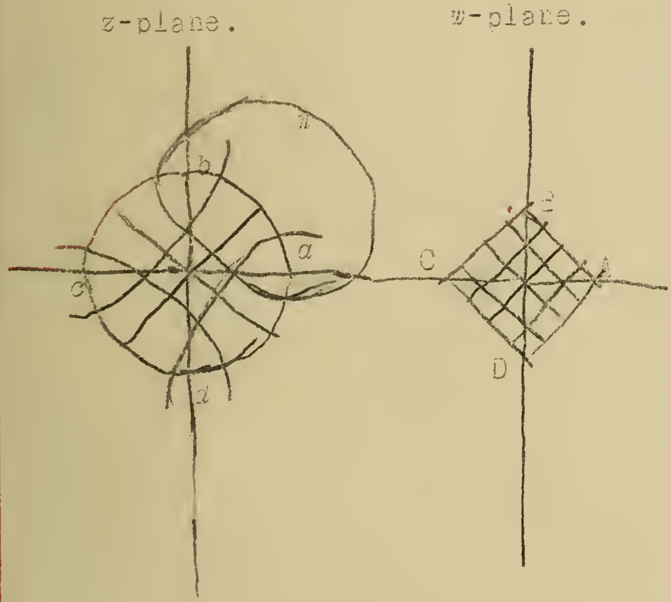
tive points, then the equation which gives the required transformation is as follows:[*]

$$w = C\int(z-a)^{\alpha-1}(z-b)^{\beta-1}\ldots\ldots(z-l)^{\lambda-1}dz + C'$$

where C and C' are arbitrary constants determinable from the position and the size of the polygon. And since a transformation can be easily found which will map the real axis into a circle we are enabled to map conformally the area within a polygon in the w-plane into a circle in the z-plane. An example of this kind is given by the equation[°]

$$w = \int_0^z \frac{dz}{\sqrt{1-z^4}} = cn^{-1}z \qquad (\text{mod} = \frac{1}{\sqrt{2}}).$$

The interior of a z-circle, center the origin and radius 1 is the conformal representation of the interior of the square in the w-plane,



with corners at A, B, C, and D, namely ± 1, $\pm i$. Denoting by L the integral $\int_0^1 \frac{dx}{\sqrt{1-x^4}}$, so that $2L$ is the length of a diagonal, the angular points of the square are D, A, B, C on the axes of reference; and these become d, a, b, c on the circumference of the circle. They correspond[1] to -1, 0, 1, ∞ on the axis of x in the representation on the half-plane. The lines parallel to AD and DC map into the lines shown in the z-plane. A complete mapping of each of these lines from $+\infty$ to $-\infty$ gives curves similar to W, which are described over and over again.

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[*] *Forsyth, Theory of Functions*, p. 541.

[°] *Ibid*, p. 545.

[1]By a further transformation.

The equation[*] $w = \int_1^z \frac{1}{z^2}(1+z^4)^{\frac{1}{2}}dz$ gives the conformal representation of the area outside of the square ABCD in the w-plane into the interior of the circle in the z-plane, center at origin and radius 1, the z origin corresponding to the infinitely distant part of the w-plane.[°]

These transformations would therefore enable one to apply Lindelöf's method to a function defined by a power series, whose circle of convergence would be inscribed in the square, and which function was known to be analytic throughout the square.

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* *Forsyth, Theory of Functions*, p. 545.

° *Ibid*, p. 545.

CHAPTER IV.

METHODS OF FORMAL CHARACTER.

18. It is possible sometimes to build up by algebraic means or otherwise new expressions formally the same as the original expressions. When this can be done for continuous regions that overlap, we have in the wider region a region in which the corresponding formal expression represents the original function. What is meant can be best shown by examples.

Let us take the function $f(z)=\log(1+z)$.

$$\log(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \ldots\ldots \qquad (1)$$

Then make the substitution $s_1(z) = -z = Z'$, and $f[s_1(z)] = \log(1-z)$. Then combine $f(z)$ and $f[s_1(z)]$ by subtracting $f[s_1(z)]$ from $f(z)$.

$$\log(1+z) - \log(1-z) = \log\frac{1+z}{1-z} = \varphi(z),$$

$$\varphi(z) \equiv 2(z + \frac{z^3}{3} + \frac{z^5}{5} + \ldots\ldots) \qquad (2).$$

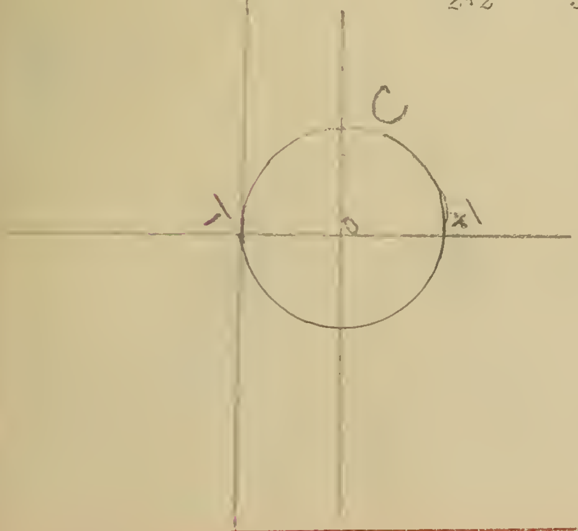
Then make the substitution

$$s_2(z) = \frac{z'}{2+z'}$$

$$\varphi[s_2(z)] = \log\left(\frac{1+\frac{z'}{2+z'}}{1-\frac{z'}{2+z'}}\right) = \log(1+z') = \psi(z')$$

and we see that $\psi(z') = f(z')$.

$$\log(1+z') = 2\left[\frac{z'}{2+z'} + \frac{1}{3}\left(\frac{z'}{2+z'}\right)^3 + \ldots\ldots\right] \qquad (3).$$



Both the series (1) and (2) converge within a circle of radius 1 about the origin as center.

But $\varphi[s_2(z)] = \psi(z') = \log(1+z')$

$= 2\left[\frac{z'}{2+z'} + \frac{1}{3}\left(\frac{z'}{2+z'}\right)^3 + \ldots\right]$, converges to the right of the line $z=-1$. So $\log(1+z')=2\left[\frac{z'}{2+z'}+\frac{1}{3}\left(\frac{z'}{2+z'}\right)^3+ \ldots\right]$ is an

analytic continuation of the series (2). For the circle C is the common region and within it the numerical values of the two expressions are equal, and (3) is analytic over the entire plane to the right of $z=-1$.

19. *Method depending on obtaining a differential equation.* Let us consider some power series $f(z)$ and take the first derivative $f'(z)$. If it is possible to equate $f'(z)$ to some algebraic expression and then take the definite integral say from 0 to z we get an analytic expression for $f(z)$ for the function over the entire portion of the plane where it is analytic, whereas the series $f(z)$ usually expresses the function over only a small portion of the plane where it is analytic namely some circle which will only cover the whole plane when the function reduces to a polynomial. Thus the analytic expression for $f(z)$ gives an analytic continuation of the series $f(z)$.

Examples. Consider the series

$$w = z + \frac{1}{2}\frac{z^3}{3} + \frac{1\cdot 3z^5}{2\cdot 4\cdot 5} + \frac{1\cdot 3\cdot 5z^7}{2\cdot 4\cdot 6\cdot 7} + \ldots\ldots \tag{1}$$

$$\frac{dw}{dz} = 1 + \frac{3}{2\cdot 3}z^2 + \frac{5\cdot 3}{5\cdot 2\cdot 4}z^4 + \frac{7\cdot 5\cdot 3}{2\cdot 4\cdot 6\cdot 7}z^6 + \ldots\ldots$$

$$= 1 + \frac{z^2}{2} + \frac{3}{8}z^4 + \frac{5}{16}z^6 + \ldots\ldots \tag{2}$$

But
$$dw = \frac{1}{\sqrt{1-z^2}}dz \tag{3}$$

$$w = f(z) = \int_0^z \frac{dz}{\sqrt{1-z^2}} = \sin^{-1}z \tag{4}$$

The series (1) expresses the function inside of a circle about the origin of radius 1 while the transcendental expression $\sin^{-1}z$ expresses the expresses the function over the entire plane if a Riemann surface is considered.

20. *Method depending on formal summing of series.* Let us consider a power series $f(z)$. If it is possible by summing this series to find an algebraic expression, then this algebraic expression is usually an analytic

continuation of the series.

The following example is a recurring series.

$$S_n = 1 + 4z + 10z^2 + 22z^3 + 46z^4 + \ldots.$$

$$pqS_n = pz + 4pz^2 + 10pz^3 + 22pz^4 + 46pz^5 + \ldots..$$

$$qz^2S_n = \qquad qz^2 + 4qz^3 + 10qz^4 + 22qz^5 + 46qz^6 + \ldots$$

$$S_n = \sum_{n=1}^{\infty} (3\cdot 2^{n-1}-2)z^{n-1}.$$

$$(1+pz+qz^2)S_n = 1+(4+p)z + (10+4p+q)z^2 + (22+10p+q)z^3 + (46+22p+10q)z^4 + \ldots..$$

Let $\quad 10 + 4p + q = 0 \qquad (1)$

$\quad 22 + 10p + 4q = 0 \qquad (2)$

$\quad 46 + 22p + 10q = 0 \qquad (3)$

The solution of (1) and (2) is $q=2p=-3$ and it satisfies (3). So $S_n = \frac{1+z}{1-3z+2z^2}$ which is analytic over the entire plane except at the points $=\frac{1}{2}$ or 1, while the series S_n converges only within a circle about the origin as center of radius $\frac{1}{2}$. So the algebraic expression S_n is a continuation of the series S_n.

21. *Method depending on the permanence of a functional equation.**

Theorem. *If* $\quad G(w_1 \ldots\ldots w_n) \qquad (1)$

is a polynomial in $(w_1 \ldots\ldots w_n)$ whose coefficients are analytic functions of z, with a common Riemann surface S and if one is then in possession of the n functions $w_i = f_i(z)$, $(i=1,2,\ldots.n)$ (2) which are all analytic at a point of S and in its neighborhood and satisfy in their neighborhood and at the point itself the polynomial $G(w_1 \ldots\ldots w_n) = 0$, (3), then each system of simultaneous analytic continuations of the functions (2) over S by a path L satisfies the functional equation (3).

L is to be a regular curve. For example differential equations are

* Osgood, *Lehrbuch der Funktionen Theorie*, pp. 389-390.

functional equations which give important examples of the application of the theorem. Let us take some linear differential equation of the second order with coefficients which are rational functions of z.

$$\frac{d^2w}{dz^2} + p(z)\frac{dw}{dz} + q(z)w = 0 \qquad (4)$$

We have given to start with the following existence theorem for the solution. Theorem*

Let $z=z_0$ be an ordinary point for the coefficients $p(z)$, $q(z)$ and further let (K) be the greatest circle around z_0 in which $p(z)$, $q(z)$ are both analytic. Then there are two linearly independent solutions of the equations w_1 and w_2, both analytic in (K). And any other solution of w which is analytic at z_0 or any other point of (K) can be expressed in the form $w=c_1w_1+c_2w_2$, where c_1 and c_2 are constants.

The theorem holds equally well if $p(z)$ and $q(z)$ are any two analytic functions in (K) which is a simply connected region which excludes $z=\infty$.

Let $w=f(z)$ be a solution of the equation where we consider $f(z)$ at first merely in a limited region in which it is analytic. Then the theorem asserts that if we set** $w_1=\frac{d^2w}{dz^2}$, $w_2=\frac{dw}{dz}$, $w_3=w$ then

$$G\ (w_1, w_2, w_3) = w_1 + p(z)w_2 + q(z)w_3$$

and every analytic continuation of $f(z)$ along a path which does not pass thru a pole of either $p(z)$ or $q(z)$, or an essential singularity, if we admit analytic functions for $p(z)$, $q(z)$, satisfies the differential equation. Accordingly the monogenic analytic function defined by these elements satisfies the differential equation everywhere in its domain.

A second example of the theorem*** is given if we consider $w_1=f_1(z)$,

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* Osgood, *Lehrbuch der Funktionen Theorie*, p. 389.

** *Ibid*, p. 390.

*** *Ibid*, p. 390-391.

$w_2=f_2(z)$ two particular solutions of the differential equation which are known in certain limited regions of the plane, thru some formula (as a power series). Then any solution $w=f(z)$ takes the form $w=c_1w_1 + c_2w_2$ in the limited region mentioned, and it is assumed that the regions of convergence of the two solutions have a certain region in common. Then the theorem asserts that whenever w_1 and w_2 can be simultaneously continued the theorem holds. Example:

Let $$w = z - \frac{z^3}{24} + \frac{7}{24}z^4 - \frac{31}{384}z^5 + \ldots.. \qquad (1)$$

then $$G(w) \equiv 16(z+1)^4w^{iv} + 96(z+1)^3w''' + 104(z+1)^2w'' + 8(z+1)w' + w$$

$$= (z+1)(z+3) \qquad (2)$$

and $$w = (z+1)^{\frac{1}{2}}\log(z+1) \qquad (3)$$

(3) is defined over the entire z-plane save at the point $z=-1$ while (1) converges only within a circle about the origin as center of radius 1. So (3) is an analytic continuation of (1) satisfying the theorem.

22. Another method which cannot be entered into here, is to set up an integral equation which the function satisfies.

CHAPTER V.

23. *Functions which cannot be continued analytically.* The fact that the representation of a function can be continued analytically beyond the domain of its initial expression should not lead us to think that all functions are of that kind. There are some functions which have no representation outside a certain boundary; such a boundary is called a natural boundary. For example the series*

$$z + \frac{z^{2!}}{2^2} + \frac{z^{3!}}{2^3} + \ldots\ldots + \frac{z^{n!}}{2} + \ldots\ldots$$

is a complete expression for a function, the natural boundary of which is a unit circle about the origin as center.

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